SPLINES, BIHARMONIC OPERATOR AND APPROXIMATE EIGENVALUES

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ABSTRACT. The biharmonic operator plays a central role in a wide array of physical models, such as elasticity theory and the streamfunction formulation of the Navier-Stokes equations. Its spectral theory has been extensively studied. In particular the one-dimensional case (over an interval) serves as the basic model of a high order Sturm-Liouville problem. The need for corresponding numerical simulations has led to numerous works. This review focuses on a discrete biharmonic calculus. The primary object of this calculus is a high-order compact discrete biharmonic operator (DBO). The DBO is constructed in terms of the discrete Hermitian derivative. The surprising strong connection between cubic spline functions (on an interval) and the DBO is recalled. In particular the kernel of the inverse of the discrete operator is (up to scaling) equal to the grid evaluation of the kernel of $\left[\left(\frac{d}{dx}\right)^4\right]^{-1}$. This fact entails the conclusion that the eigenvalues of the DBO converge (at an "optimal" $O(h^4)$ rate) to the continuous ones. Another consequence is the validity of a *comparison principle.* It is well known that there is no maximum principle for the fourth-order equation. However, a positivity result is recalled, both for the continuous and the discrete biharmonic equation, claiming that in both cases the kernels are order preserving.

1. INTRODUCTION

The aim of this paper is to review some recent results concerning surprising connections between cubic spline functions and a discrete approximation to the one-dimensional biharmonic operator.

The operator $\left(\frac{d}{dx}\right)^4$ on the interval $[0,1]$ is certainly the simplest conceivable example of a fourth-order elliptic one-dimensional operator. As such, its spectral theory is very well understood [10, Chapter 5] or [14]. In classical terminology, its study is labeled as a "fourth-order Sturm-Liouville theory". More generally, one can consider the spectral structure of operators of the form $\left(\frac{d}{dx}\right)^4 + \frac{d}{dx}\left(A(x)\frac{d}{dx}\right) + B(x)$. For such operators it was proved in [8] that the isospectral set (of coefficients $A(x)$, $B(x)$) is an infinite-dimensional real-analytic manifold (provided the spectrum is simple).

Fourth-order elliptic operators , and particularly the biharmonic operator, play a significant role in a variety of physical models, such as elasticity theory, the streamfunction formulation of the Navier-Stokes equations [4] or quasi-geostrophic ocean flows [21].

There is a vast literature devoted to a variety of discrete approximations to the solutions of fourth-order equations. Since in this review we focus on the *one-dimensional eigenvalue problem*, we shall just refer to studies that are closely related to this issue.

The numerical evaluation of eigenvalues has been a fundamental goal in the development of numerical analysis, and as such the subject of numerous studies. As representative examples we can mention the "Shannon-type" sampling method in [7] , the "matrix methods" in [24], the finite element methods in [2] and a 7-diagonal finite difference method in [9]. The aim of these works was to capture the eigenvalues of the continuous operator by a suitable approximation procedure.

Here we single out a "high order" approximation of the eigenvalues of the one-dimensional biharmonic operator. However, the approach adapted here is based on a "discrete elliptic theory", as has recently been expounded in [5]. It involves the construction of discrete elliptic operators that can be shown to possess the classical elliptic properties, such as coercivity and regularity. The fundamental discrete operator here is the discrete biharmonic operator (DBO) δ_x^4 (2.9). The idea is to consider this DBO as a finite-dimensional *operator* approximation to $\left(\frac{d}{dx}\right)^4$ and to conclude that the eigenvalues of the latter are limits (under mesh refinement) of the eigenvalues of the former.

Date: January 14, 2025.
2010 *Mathematics Subject Classification.* Primary 34L16; Secondary 34B24, 41A15,65L10.
Key words and phrases. cubic splines, Hermitian derivative, discrete biharmonic operator, eigenvalues, Green's kernel.
It is a pleasure to thank Jean-Pierre Croisille, Dalia Fishelov and Guy Katriel for very fruitful discussions.

It is well known that the convergence of finite-dimensional approximations to an infinite-dimensional, unbounded differential operator, does not entail the convergence of the respective spectra. Therefore, a deeper connection between the discrete and continuous operators is required. The bridge between the two operators is achieved by using the classical cubic spline functions.

A basic tool of the discrete elliptic calculus is the *discrete Hermitian derivative* on an interval, that gives a fourth-order accurate approximation to the derivative of a smooth function. It has been the cornerstone in the construction of the DBO [15] and its extension to the full fourth-order Sturm-Liouville problem [5].

The structure of this review paper is as follows.

In Section 2 we recall the definitions of the discrete finite difference operators, and in particular introduce the Hermitian derivative and the discrete biharmonic operator δ_x^4.

In Subsection 3.1 we recall the basic (classical) construction of cubic spline functions on an interval.

In Subsection 3.2 we first state the equality of the Hermitian derivative and the derivative of the interpolating cubic spline. This is a fundamental fact connecting the two non-local fourth-order approximations of the derivative.

The connection between the discrete biharmonic operator δ_x^4 and the interpolating cubic spline function is then presented. It is in fact the primary goal of this paper. Recall that cubic splines are C^2 functions, with finite jumps of the third-order derivatives at grid points. The basic result here (Proposition 3.8) is that the sizes of these jumps are determined by the DBO acting on the grid values. We have not been able to locate such a result in the "spline literature", perhaps due to the fact that the DBO is not explicitly considered there.

This connection enables us to recall, in Section 4 , positivity results for the continuous and discrete fourth-order operators (see Proposition 4.1 and Proposition 4.3). Recall that there is no maximum-minimum principle for the fourth-order operator, so that the "order-preserving" property could serve as a substitute in some cases.

In Section 5 we first give the explicit form of the kernel (Green's function) of the continuous operator. In the first instance, this kernel acts in $L^2(0,1)$. We then present its extension to the negative Sobolev space $H^{-2}(0,1)$. This space includes all finite measures, and in particular all grid functions (identified as linear combinations of delta functions at the grid points). Using the connection to cubic spline functions a remarkable result emerges: the discrete resolvent (namely, the kernel of $(\delta_x^4)^{-1}$) is just the grid evaluation of the continuous kernel, up to scaling. Indeed, this can be viewed as an alternative, very natural, definition of the compact discrete biharmonic operator.

Finally, Section 6 is concerned with the subject matter of this review, namely, the relationship between the *eigenvalues* of the continuous and discrete operators. The connection between the discrete and continuous kernels implies that the discrete eigenvalues are actually obtained by a "Nyström method" [26].

It is the aforementioned connection between the kernels that entails not only the mere convergence of the discrete eigenvalues to the continuous ones, but also that this convergence proceeds at an "optimal" fourth-order rate (Theorem 6.13). While we do not give full details of the proof, we indicate how this result is obtained by combining two ingredients:

- A suitable adaptation (Lemma 6.11) of a more general abstract convergence theorem [18, 20]. However, we have chosen to provide a self-contained, much simpler proof, that builds on the analytic theory of finite-dimensional perturbations, as expounded in Kato's classical book [17].
- The ability to estimate the differences of the continuous and the discrete operators, including the optimal rates of convergence, in terms of differences of their respective kernels, see Proposition 6.4.

2. SETUP and DEFINITION OF THE DISCRETE OPERATORS

We equip the interval $\Omega = [0,1]$ with a uniform grid

$$x_j = jh, \quad 0 \le j \le N, \quad h = \frac{1}{N}.$$

The approximation is carried out by grid functions $\mathfrak{v}$ defined on $\{x_j,\, 0 \le j \le N\}$. The space of these grid functions is denoted by l_h^2. For their components we use either $\mathfrak{v}_j$ or $\mathfrak{v}(x_j)$.

For every smooth function $f(x)$ we define its associated grid function

$$f_j^* = f(x_j), \quad 0 \le j \le N. \tag{2.1}$$

The discrete l_h^2 scalar product is defined by

$$(\mathfrak{v}, \mathfrak{w})_h = h \sum_{j=0}^{N} \mathfrak{v}_j \mathfrak{w}_j,$$

and the corresponding norm is

$$|\mathfrak{v}|_h^2 = h\sum_{j=0}^{N}\mathfrak{v}_j^2. \tag{2.2}$$

For linear operators $\mathcal{A} : l_h^2 \to l_h^2$ we use $|\mathcal{A}|_h$ to denote the operator norm.

The discrete sup-norm is

$$|\mathfrak{v}|_\infty = \max_{0\le j\le N}\{|\mathfrak{v}_j|\}. \tag{2.3}$$

The discrete homogeneous space of grid functions is defined by

$$l_{h,0}^2 = \{\mathfrak{v},\ \mathfrak{v}_0 = \mathfrak{v}_N = 0\}. \tag{2.4}$$

Given $\mathfrak{v} \in l_{h,0}^2$ we introduce the basic (central) finite difference operators

$$\begin{aligned}(\delta_x\mathfrak{v})_j &= \frac{1}{2h}(\mathfrak{v}_{j+1} - \mathfrak{v}_{j-1}), \quad 1 \le j \le N-1,\\ (\delta_x^2\mathfrak{v})_j &= \frac{1}{h^2}(\mathfrak{v}_{j+1} - 2\mathfrak{v}_j + \mathfrak{v}_{j-1}), \quad 1 \le j \le N-1,\end{aligned} \tag{2.5}$$

The cornerstone of our approach to finite difference operators is the introduction of the **Hermitian derivative** [5] of $\mathfrak{v} \in l_{h,0}^2$, that will replace δ_x. It will serve not only in approximating (to fourth-order of accuracy) first-order derivatives, but also as a fundamental building block in the construction of finite difference approximations to higher-order derivatives.

First, we introduce the "Simpson operator"

$$(\sigma_x\mathfrak{v})_j = \frac{1}{6}\mathfrak{v}_{j-1} + \frac{2}{3}\mathfrak{v}_j + \frac{1}{6}\mathfrak{v}_{j+1}, \quad 1 \le j \le N-1. \tag{2.6}$$

Note the operator relation (valid in $l_{h,0}^2$)

$$\sigma_x = I + \frac{h^2}{6}\delta_x^2, \tag{2.7}$$

so that σ_x is an "approximation to the identity".

The Hermitian derivative $\mathfrak{v}_x$ is now defined by

$$(\sigma_x\mathfrak{v}_x)_j = (\delta_x\mathfrak{v})_j, \quad 1 \le j \le N-1. \tag{2.8}$$

Remark 2.1. *In the definition* (2.8)*, the values of* $(\mathfrak{v}_x)_j,\ j = 0, N,$ *need to be provided , in order to make sense of the left-hand side (for* $j = 1, N-1$*). If not otherwise specified, we shall henceforth assume that* $\mathfrak{v}_x \in l_{h,0}^2$*, namely*

$$(\mathfrak{v}_x)_0 = (\mathfrak{v}_x)_N = 0.$$

In particular, the linear correspondence $l_{h,0}^2 \ni \mathfrak{v} \to \mathfrak{v}_x \in l_{h,0}^2$ *is well defined, but not onto, since* δ_x *has a non-trivial kernel.*

The discrete biharmonic (DBO) operator is given by (for $\mathfrak{v}$, $\mathfrak{v}_x \in l_{h,0}^2$),

$$\delta_x^4\mathfrak{v} = \frac{12}{h^2}[\delta_x\mathfrak{v}_x - \delta_x^2\mathfrak{v}]. \tag{2.9}$$

The truncation error of the DBO is $O(h^4)$ at internal points but only $O(h)$ at near-boundary points [4, Proposition 10.8]. However, the full ("optimal") fourth-order accuracy is achieved by its inverse (see Equation (2.15) below). This is a fundamental fact in the present study.

We next introduce a fourth-order replacement to the operator δ_x^2 (see [4, Equation (10.50)(c)]),

$$(\widetilde{\delta_x^2}\mathfrak{v})_j = 2(\delta_x^2\mathfrak{v})_j - (\delta_x\mathfrak{v}_x)_j, \quad 1 \le j \le N-1. \tag{2.10}$$

Note that, in accordance with Remark 2.1 the operator $\widetilde{\delta_x^2}$ is defined on grid functions $\mathfrak{v} \in l_{h,0}^2$, so that also $\mathfrak{v}_x \in l_{h,0}^2$.

The connection between the two difference operators for the second-order derivative is given by

$$-\widetilde{\delta_x}^{\,2} = -\delta_x^2 + \frac{h^2}{12}\delta_x^4. \tag{2.11}$$

Remark 2.2. *Clearly the operators δ_x, δ_x^2, δ_x^4 depend on h, but for notational simplicity this dependence is not explicitly indicated.*

The fact that the biharmonic discrete operator δ_x^4 is positive (in particular symmetric) is proved in [4, Lemmas 10.9, 10.10]. Therefore its inverse $\left(\delta_x^4\right)^{-1}$ is also positive. In fact, it satisfies a strong coercivity property, that is also established in the aforementioned reference.

Another way (closer to the Hermitian approach) to define the finite-difference operators $\widetilde{\delta_x}^{\,2}$ and δ_x^4 is in terms of a "polynomial approach" [4, Section 10.3], as follows. Let $q(x)$ be a fourth-order polynomial such that

$$q(x_j) = \mathfrak{v}_j,\ q(x_{j\pm1}) = \mathfrak{v}_{j\pm1},\ q'(x_{j\pm1}) = (\mathfrak{v}_x)_{j\pm1}.$$

Then

$$(\widetilde{\delta_x}^{\,2}\mathfrak{v})_j = q''(x_j),\ (\delta_x^4\mathfrak{v})_j = q^{(4)}(x_j). \tag{2.12}$$

The discrete biharmonic operator gives a very accurate approximation to the continuous one ("optimal 4-th order accuracy") , as seen in the following claim [4, Theorem 10.19] .

Claim 2.3. *Let $f(x) \in C^4(\Omega),\ \Omega = [0,1]$. Let $u(x)$ satisfy*

$$\left(\frac{d}{dx}\right)^4 u(x) = f(x), \tag{2.13}$$

subject to homogeneous boundary conditions

$$u(0) = \frac{d}{dx}u(0) = u(1) = \frac{d}{dx}u(1) = 0. \tag{2.14}$$

Then

$$|u^* - (\delta_x^4)^{-1}f^*|_\infty = O(h^4). \tag{2.15}$$

Remark 2.4. *The "$O(h^4)$" here means that there exists a constant $C > 0$, depending only on f, such that for all integers $N > 1$,*

$$|u^* - (\delta_x^4)^{-1}f^*|_\infty \le Ch^4, \quad h = \frac{1}{N}.$$

Observe that the grid functions in this estimate are defined on the grid of (the variable) mesh size h.

3. SPLINES , HERMITIAN DERIVATIVES and the DISCRETE BIHARMONIC OPERATOR

3.1. THE BASIC SETUP for CUBIC SPLINES.

In this subsection we recall the basic facts about cubic splines that will be essential in this study.

As in Section 2 we consider the interval $\Omega = [0,1]$ with a uniform grid

$$x_j = jh, \quad 0 \le j \le N, \quad h = \frac{1}{N}.$$

We fix a vector $\mathfrak{f} = \{\mathfrak{f}_j\}_{j=0}^N$ so that $\mathfrak{f}_0 = \mathfrak{f}_N = 0$, namely $\mathfrak{f} \in l^2_{h,0}$ (see (2.4)), and consider the family

$$\mathcal{A} = \left\{u \in H_0^2(\Omega), \quad u_j = \mathfrak{f}_j, \quad j = 0,1,...,N\right\}.$$

The space $H_0^2(\Omega)$ is the space of functions having first and second (distrbutional) derivatives in $L^2(\Omega)$ and vanishing, with their first-order derivatives, at the endpoints.

It is well known that the norm in $H_0^2(\Omega)$ can be defined by

$$\|u\|^2_{H_0^2(\Omega)} = \int_0^1 |u''(x)|^2 dx,$$

and we shall refer henceforth to this norm.

We consider the functional

$$I(u) = \int_0^1 |u''(x)|^2 dx, \quad u \in H_0^2(\Omega).$$

We are interested in a minimizer for this functional, restricted to $\mathcal{A}$.

Claim 3.1. *The functional has a unique minimizer on $\mathcal{A}$, which we designate as $s_{\mathfrak{f}}$,*

$$I(s_{\mathfrak{f}}) < I(g), \quad s_f \neq g \in \mathcal{A}.$$

The proof of this classical fact can be worked out by standard methods of the calculus of variations [13, 23]. A purely algebraic proof can be found in [1, Theorem 3.4.3] or [11, Chapter IV, Cubic Spline Interpolation]. The reader can also find the proof of the following claim in these latter references.

Claim 3.2. (1) *$s_{\mathfrak{f}}$ is a cubic polynomial in each interval $[x_j, x_{j+1}]$, $\quad j = 0, 1, ..., N-1$.*
(2) *$s_{\mathfrak{f}} \in C_0^2(\Omega)$.*
(3) *The previous two properties , supplemented by the constraints $s_{\mathfrak{f}}(x_j) = \mathfrak{f}_j, \quad j = 1, ..., N-1$, and $s_{\mathfrak{f}}(x_0) = s'_{\mathfrak{f}}(x_0) = s_{\mathfrak{f}}(x_N) = s'_{\mathfrak{f}}(x_N) = 0$ determine $s_{\mathfrak{f}}$ uniquely.*

Definition 3.3. *The function $s_{\mathfrak{f}}$ is called the ("type I")* ***cubic spline*** *corresponding to the constraints*

$$s_{\mathfrak{f}}(x_j) = \mathfrak{f}_j, \quad j = 1, ..., N-1, \ s_{\mathfrak{f}}(x_0) = s'_{\mathfrak{f}}(x_0) = s_{\mathfrak{f}}(x_N) = s'_{\mathfrak{f}}(x_N) = 0.$$

Claim 3.4. *Consider the vectors $\mathfrak{f} = \{\mathfrak{f}_j\}_{j=0}^N$ such that $\mathfrak{f}_0 = \mathfrak{f}_N = 0$, namely $\mathfrak{f} \in l^2_{h,0}$ (see (2.4)). Then the map $\mathfrak{f} \hookrightarrow s_{\mathfrak{f}} \in H_0^2(\Omega)$ is one-to-one and linear.*

Remark 3.5. *A positivity property of the cubic spline is stated in Corollary 4.4 below.*

3.2. **CUBIC SPLINES MEET the DISCRETE BIHARMONIC OPERATOR.** We use the notation of Section 2.

Let $\mathfrak{u} \in l^2_{h,0}$ be a grid function vanishing at the endpoints and let $s_{\mathfrak{u}} \in H_0^2(\Omega)$ be the corresponding spline function (Claim 3.4).

We use interchangeably the notation $\mathfrak{u}_j = \mathfrak{u}(x_j)$.

Let $\mathfrak{u}_x$ be the Hermitian derivative of $\mathfrak{u}$, and we set at the endpoints

$$\mathfrak{u}_x(x_0) = s'_{\mathfrak{u}}(x_0) = 0, \mathfrak{u}_x(x_N) = s'_{\mathfrak{u}}(x_N) = 0. \tag{3.1}$$

The remarkable fact about the equality of the Hermitian derivative and the derivative of the spline function is stated in the following proposition (see [6] for the proof).

Proposition 3.6. *For all interior nodes, $s'_{\mathfrak{u}}(x_j) = \mathfrak{u}_x(x_j), \quad 1 \leq j \leq N-1$.*

In addition to $\mathfrak{u} \in l^2_{h,0}$, let $\mathfrak{v} \in l^2_{h,0}$ be a grid function vanishing at the endpoints and let $s_{\mathfrak{v}}$ be the corresponding spline function. At the endpoints we impose again the boundary conditions (3.1).

Claim 3.7. *The map $(\mathfrak{u}, \mathfrak{v}) \to \int_0^1 s''_{\mathfrak{u}}(x) s''_{\mathfrak{v}}(x) dx$ is a scalar product on $l^2_{h,0}$.*

Proof. In view of Claim 3.4 the map is bilinear. Furthermore , if $\int_0^1 |s''_{\mathfrak{u}}(x)|^2 dx = 0$, then $s''_{\mathfrak{u}} \equiv 0$ and since $s_{\mathfrak{u}} \in H_0^2$ it follows that also $s_{\mathfrak{u}} \equiv 0$, which implies $\mathfrak{u} = 0$. □

We denote by $\delta_x^4 \mathfrak{u}$ the DBO action on $\mathfrak{u}$ (see (2.9)). The profound connection between $\delta_x^4 \mathfrak{u}$ and the derivatives of $s_{\mathfrak{u}}$ is given in the following proposition (see [6] for the proof).

Proposition 3.8. *Let $\mathfrak{u}, \mathfrak{u}_x, \mathfrak{v}, \mathfrak{v}_x \in l^2_{h,0}$.*

- *The discrete scalar product of $\delta_x^4 \mathfrak{u}$ and $\mathfrak{v}$ satisfies*

$$(\delta_x^4 \mathfrak{u}, \mathfrak{v})_h = \int_0^1 s''_{\mathfrak{u}}(x) s''_{\mathfrak{v}}(x) dx. \tag{3.2}$$

- *The jump of the third order derivatives of the cubic splines at the nodes is given by*

$$s_{\mathfrak{u}}'''(x_j^+) - s_{\mathfrak{u}}'''(x_j^-) = h(\delta_x^4 \mathfrak{u})_j, \quad j = 1, \dots, N-1. \tag{3.3}$$

Remark 3.9. *In the literature (e.g.* [1, 11] *one can find various expressions for the jump of the third order derivatives of the cubic spline. However Equation* (3.3) *provides a new expression, that can be interpreted as a "fourth-order derivative" of the function at the node.*

3.3. COMPARING THE FEM and DBO APPROACHES to

$$\left(\frac{d}{dx}\right)^4 u(x) = f(x).$$

. The relation of the DBO to cubic spline functions, as expressed in Proposition 3.8, raises the question about the connection between the "discrete functional calculus" and the finite-element approaches to the approximation of the continuous biharmonic equation. In the following discussion we clarify the distinction between them.

If the cubic splines are taken as "basis functions", the variational formulation via the finite-element methodology [19, 23] means that we look for a grid function $\mathfrak{u}$ that satisfies

$$\int_0^1 s_{\mathfrak{u}}''(x) s_{\mathfrak{v}}''(x)dx = \int_0^1 s_{f^*}(x) s_{\mathfrak{v}}(x)dx, \quad \text{for all grid functions } \mathfrak{v} \in l^2_{h,0}. \tag{3.4}$$

On the other hand, the discrete functional approach employed here implies that we look for a grid function $\mathfrak{u}$ that satisfies

$$(\delta_x^4 \mathfrak{u}, \mathfrak{v})_h = (f^*, \mathfrak{v})_h, \quad \text{for all grid functions } \mathfrak{v} \in l^2_{h,0}. \tag{3.5}$$

While the left-hand sides in Equations (3.4) and (3.5) are equal (Proposition 3.8), this is in general not true for the right-hand sides. This shows that, in spite of the connection between the DBO and cubic splines expounded above, the DBO scheme is not equivalent to the FEM based on these splines.

4. POSITIVITY

It is well known that there is (in general) no maximum principle for elliptic partial differential operators of order higher than two. For the biharmonic equation in multi-dimensional domains there exist versions of the principle that involve estimates of the gradient of the solution, see [22] and references therein. Under Dirichlet boundary conditions (the only ones considered here) the *preservation of positivity property* means that $\Delta^2 u \geq 0 \Rightarrow u \geq 0$. It is actually a *property of the domain.* The maximum principle implies preservation of positivity but of course not vice versa. In the one-dimensional case a general study of linear differential inequalities is given in [25]. In the multi-dimensional case (excluding the one-dimensional case) we refer to [16] and references therein.

In our one-dimensional case we have the following proposition. Besides being of interest in its own right, it motivates the requirement that discrete approximations possess the same property (satisfied by the DBO, see Proposition 4.3 below). The proof of this property in the discrete case, in turn, implies a positivity property of cubic splines (Corollary 4.4 below).

Proposition 4.1. *Let*

$$\left(\frac{d}{dx}\right)^4 u(x) = f(x),$$

where $u \in H^4(\Omega) \cap H_0^2(\Omega)$*. Then the following comparison principle holds.*

If $f(x) \geq 0,\ x \in \Omega$*, then also* $u(x) \geq 0,\ x \in \Omega$*.*

Proof. Suppose to the contrary that for some $y \in (0, 1)$ we have $u(y) < 0$. We can assume that y is a minimum point for u, so that

$$u'(y) = 0,\ u''(y) \geq 0.$$

Since u' vanishes at the endpoints, we infer that there are points

$$\xi \in (0, y),\ \eta \in (y, 1),$$

such that

$$u''(\xi) = u''(\eta) = 0.$$

Let

$$a = \inf\Big\{\xi \in \Omega,\ u''(\xi) = 0\Big\},$$
$$b = \sup\Big\{\eta \in \Omega,\ u''(\eta) = 0\Big\}. \tag{4.1}$$

Consider the function $v(x) = u''(x)$. It satisfies in the interval $[a, b]$ the inequality

$$v''(x) = f(x) \geq 0,$$

as well as $v(a) = v(b) = 0$ and $v(y) \geq 0$.

The standard maximum principle now yields

$$v(x) \equiv 0,\ x \in [a, b],$$

hence also $u'(x) \equiv u'(y) = 0$, $x \in [a, b]$.

If $a > 0$ we get a contradiction since there is a point $\xi \in (0, a)$ with $u''(\xi) = 0$. Similarly if $b < 1$. We conclude that $u'(x) \equiv 0$, $x \in [0, 1]$, hence $u(x) \equiv u(y) < 0$, $x \in [0, 1]$. However this contradicts the boundary condition $u(0) = u(1) = 0$.

□

Remark 4.2. *In Section 5 below we derive an expression for the resolvent kernel* (5.3)*. Since it is easy to see that the kernel is nonnegative, we obtain another proof of Proposition 4.1.*

4.1. **POSITIVITY of the DISCRETE BIHARMONIC OPERATOR.** We now show that the same positivity property holds also for the discrete biharmonic operator.

Proposition 4.3. *Let*

$$\delta_x^4 \mathfrak{u} = \mathfrak{f},$$

where $\mathfrak{u}$, $\mathfrak{u}_x \in l^2_{h,0}$. *Then the following comparison principle holds.*

If $\mathfrak{f}_j \geq 0,\ 0 \leq j \leq N$, *then also* $\mathfrak{u}_j \geq 0,\ 0 \leq j \leq N$.

Proof. Suppose to the contrary that $\mathfrak{u}_{j_0} < 0$ for some index $1 \leq j_0 \leq N - 1$.

Let $s_\mathfrak{u} \in C_0^2(\Omega)$ be the corresponding spline function. Since $s_\mathfrak{u}(x_{j_0}) = \mathfrak{u}_{j_0} < 0$ it follows that there exists a minimum point $y \in \Omega$ so that

$$s_\mathfrak{u}(y) = \min\{s_\mathfrak{u}(x),\ x \in \Omega\} < 0.$$

We have

$$s'_\mathfrak{u}(y) = 0,\ \ s''_\mathfrak{u}(y) \geq 0. \tag{4.2}$$

Since $s'_\mathfrak{u}$ vanishes at the endpoints, we infer that there are points

$$\xi \in (0, y),\ \eta \in (y, 1),$$

such that

$$s''_\mathfrak{u}(\xi) = u''(\eta) = 0.$$

Let

$$a = \inf\Big\{\xi \in \Omega,\ s''_\mathfrak{u}(\xi) = 0\Big\},$$
$$b = \sup\Big\{\eta \in \Omega,\ s''_\mathfrak{u}(\eta) = 0\Big\}. \tag{4.3}$$

Let $w(x) = s''_\mathfrak{u}(x)$. The function w is continuous and linear in grid intervals. In view of Proposition 3.8 we get, in the sense of distributions,

$$w'' = h\sum_{j=1}^{N-1} \mathfrak{f}_j \delta_{x_j} \geq 0, \tag{4.4}$$

where δ_y is the Dirac measure at y.

Since $w(a) = w(b) = 0$, the standard maximum principle yields

$$w(x) \equiv 0,\ x \in [a, b],$$

hence

$$s'_{\mathfrak{u}}(x) \equiv s'_{\mathfrak{u}}(y) = 0,\ x \in [a,b],$$

and in particular $s'_{\mathfrak{u}}(a) = s'_{\mathfrak{u}}(b) = 0$.

As in the proof of Proposition 4.1 we conclude that $a = 0$ and $b = 1$, and therefore

$$s_{\mathfrak{u}}(x) \equiv s_{\mathfrak{u}}(y) < 0,\ x \in [0,1],$$

which is a contradiction to the boundary conditions.

□

Corollary 4.4. *Let $\mathfrak{u}$ satisfy the conditions of Proposition 4.3. Let $s_{\mathfrak{u}}$ be the corresponding spline function. Then*

$$s_{\mathfrak{u}}(x) \geq 0, \quad x \in [0,1].$$

Proof. The assumption that there exists a point $y \in (0,1)$ such that $s_{\mathfrak{u}}(y) < 0$ leads to a contradiction; this follows from the proof of Proposition 4.3 . □

5. THE CONTINUOUS and DISCRETE RESOLVENT KERNEL

The operator $\mathcal{L} = d^4/dx^4$, with homogeneous boundary conditions ($\phi \in D(\mathcal{L}) \Rightarrow \phi(0) = \phi'(0) = \phi(1) = \phi'(1) = 0$) is positive definite (in particular self adjoint) with domain $D(\mathcal{L}) = H^4([0,1]) \cap H_0^2([0,1]$. We now consider the kernel of $\mathcal{L}^{-1}$, namely, Green's function of the biharmonic problem

$$\mathcal{L}u = \Big(\frac{d}{dx}\Big)^4 u(x) = f(x), \tag{5.1}$$

where $u \in H^4(\Omega) \cap H_0^2(\Omega)$. A standard computation leads to the following

Claim 5.1. *The solution of* (5.1) *is given by*

$$u(x) = \int_0^1 K(x,y) f(y) dy, \tag{5.2}$$

where

$$K(x,y) = \begin{cases} \frac{1}{6}(1-x)^2 y^2 [2x(1-y) + x - y], & y < x \\ \frac{1}{6}x^2(1-y)^2[2y(1-x) + y - x], & x < y \end{cases}. \tag{5.3}$$

Proof. By the general theory, we verify that in the sense of distributions, for each fixed y, as a function of x,

$$\Big(\frac{d}{dx}\Big)^4 K(x,y) = \delta_y,$$

where δ_y is the Dirac measure at y. In addition, $K(x,y)$ is symmetric in x, y and satisfies the homogeneous boundary conditions (as a function of x). □

5.1. EXTENDING the KERNEL to $H^{-2}(\Omega)$.

The domain of $\left(\frac{d}{dx}\right)^4$ (as a self-adjoint operator in $L^2(\Omega)$, subject to homogeneous boundary conditions) is $H_0^2(\Omega) \cap H^4(\Omega)$. When extended (in the sense of distributions) to $H_0^2(\Omega)$, it maps it to its dual $H^{-2}(\Omega)$ [13, Chapter 5]. On the other hand, the general theory (or a direct inspection of the expression (5.3)) ensures that, for every fixed $x \in \Omega$, we have $K(x,\cdot) \in H_0^2(\Omega)$. It follows that Equation (5.2) can be extended to all $u \in H_0^2(\Omega)$ (or, alternatively, to all $f \in H^{-2}(\Omega)$) as

$$u(x) = < K(x,y), f(y) >, \tag{5.4}$$

where $< \cdot, \cdot >$ is the $\Big(H_0^2(\Omega),\ H^{-2}(\Omega)\Big)$ coupling.

As follows from Equation (3.3), the action of the operator $\left(\frac{d}{dx}\right)^4$ on splines is given by a combination of Dirac delta-functions at the nodes x_j, namely, it can be written as an equality of grid functions

$$\Big(\frac{d}{dx}\Big)^4 s_{\mathfrak{u}} = h\delta_x^4 \mathfrak{u}.$$

The right-hand side in this equation is a finite measure, and we recall that, owing to the Sobolev embedding theorem, all finite measures are contained in $H^{-2}(\Omega)$.

Thus, Equation (5.4) takes here the form

$$\mathfrak{u}_j = h \sum_{i=1}^{N-1} K(x_i, x_j)(\delta_x^4 \mathfrak{u})_i, \quad j = 1, 2, \ldots, N-1. \tag{5.5}$$

Corollary 5.2. *The discrete operator* $(\delta_x^4)^{-1} : l_{h,0}^2 \to l_{h,0}^2$ *is represented by a matrix* $\left\{K_{i,j}^h\right\}_{1 \le i,j \le N-1}$, *explicitly given by*

$$K_{i,j}^h = hK(x_i, x_j), \quad 1 \le i, j \le N-1, \tag{5.6}$$

where $K(x, y)$ *is the resolvent kernel of* $\left(\frac{d}{dx}\right)^4$, *as in Equation* (5.3).

6. CONTINUOUS and DISCRETE EIGENVALUES

In this section we reach the main purpose of this paper, namely, reviewing the convergence of the discrete eigenvalues (of the DBO) to the eigenvalues of the continuous operator $\left(\frac{d}{dx}\right)^4$. Continuing the discussion in Subsection 3.3, it is important to make the distinction between our "discrete functional calculus" approach to that of the closely related finite-element approach. For the latter, we refer to the extensive survey [3].

In the finite-element methodology, given a mesh size $h = \frac{1}{N}$, an eigenvalue μ_h and the associated eigenfunction $s_{\mathfrak{u}_h}(x)$ are obtained by the equation (compare Eq. (3.4))

$$\int_0^1 s''_{\mathfrak{u}_h}(x) s''_{\mathfrak{v}}(x) dx = \mu_h \int_0^1 s_{\mathfrak{u}_h}(x) s_{\mathfrak{v}}(x) dx, \quad \text{for all grid functions } \mathfrak{v} \in l_{h,0}^2. \tag{6.1}$$

On the other hand, in the approach employed here we look for an eigenvalue λ_h and a grid function $\mathfrak{u}_h \in l_{h,0}^2$ that satisfy

$$(\delta_x^4 \mathfrak{u}_h, \mathfrak{v})_h = \lambda_h (\mathfrak{u}_h, \mathfrak{v})_h, \quad \text{for all grid functions } \mathfrak{v} \in l_{h,0}^2. \tag{6.2}$$

While the left-hand sides are equal, in view of Proposition 3.8, this is not true in general for the right-hand sides. For this reason, we cannot invoke the well-developed theory of spectral approximation in the finite-element framework [3] in order to obtain the convergence of eigenvalues in our setup.

6.1. THE CONTINUOUS OPERATOR.

The operator $\mathcal{L}$, introduced in Section 5, has a compact resolvent, and the kernel K of $\mathcal{L}^{-1}$ is given in Claim 5.1. The spectrum of $\mathcal{L}$ consists of an increasing sequence of positive simple eigenvalues, which we designate as $\{0 < \lambda_1 < \lambda_2 < ... < \lambda_k < ...\}$.

Since these eigenvalues play an important role in the sequel, we provide below the details of their evaluation, repeating the proof of [10, Lemma 5.5.4].

Let $\phi \in H^4([0,1]) \cap H_0^2([0,1])$ be a real eigenfunction

$$\frac{d^4}{dx^4}\phi = \lambda\phi, \quad \lambda \in \{0 < \lambda_1 \le ... \le \lambda_k ...\}.$$

Clearly, this function must be of the form

$$\phi(x) = A\cos(\beta x) + B\sin(\beta x) + C\cosh(\beta x) + D\sinh(\beta x), \tag{6.3}$$

where β is real and $\beta^4 = \lambda$.

The conditions $\phi(0) = \phi'(0) = 0$ clearly imply

$$A = -C, \quad B = -D,$$

and $\phi(1) = 0$ yields

$$A(\cos\beta - \cosh\beta) = -B(\sin\beta - \sinh\beta). \tag{6.4}$$

The remaining condition $\phi'(1) = 0$ yields

$$-B(\cos\beta - \cosh\beta) = A(-\sin\beta - \sinh\beta).$$

Multiplying the two equations and invoking standard identities we get

$$\cos\beta\cosh\beta = 1, \tag{6.5}$$

which is to be considered as the equation determining the discrete eigenvalues.

Changing $\beta \to -\beta$ we can keep A, C unmodified but reverse the signs of B, D. It therefore follows that for $-\beta < 0$ (solution of (6.5)) we get the same eigenfunction (6.3) as for $\beta > 0$, and we can consider only positive β.

We therefore get the full set of eigenfunctions (for $\beta > 0$ solving (6.5)),

$$\phi(x) = A\cos(\beta x) + B\sin(\beta x) - A\cosh(\beta x) - B\sinh(\beta x), \tag{6.6}$$

where A, B satisfy (6.4).

In order to estimate the location of the eigenvalues it therefore suffices to consider the positive solutions of (6.5). The following claim is easy to verify.

Claim 6.1. *Equation* (6.5) *has a sequence of positive solutions as follows.*

$$\begin{cases} \beta_0 \in (3\pi/2, 2\pi), \\ \beta_k^{(1)} \in (2k\pi, (2k+1/2)\pi), \quad k = 1, 2, ... \\ \beta_k^{(2)} \in ((2k+3/2)\pi, (2(k+1)\pi), \quad k = 1, 2, ... \end{cases} \tag{6.7}$$

The corresponding eigenvalues $\lambda_0 = \beta_0^4$, $\lambda_k^{(1)} = (\beta_k^{(1)})^4$, $\lambda_k^{(2)} = (\beta_k^{(2)})^4$ *of* $\mathcal{L}$ *are all simple.*

We denote by

$$\{\phi_1, ..., \phi_k ...\}$$

the orthonormal set of the associated eigenfunctions.

6.2. DISCRETE EIGENVALUES–ENSEMBLE ESTIMATES.

We simplify the notation above and denote by $\{0 < \lambda_1 < \lambda_2 < \ldots < \lambda_k < \ldots\}$ the (infinite) sequence of eigenvalues of $\mathcal{L} = \left(\frac{d}{dx}\right)^4$.

Given $h = \frac{1}{N}$, let

$$\Lambda_h = \{0 < \lambda_{h,1} \le \lambda_{h,2} \le \ldots \le \lambda_{h,N-1}\}$$

be the finite sequence of eigenvalues of δ_x^4.

We denote by Γ the sum

$$\Gamma = \sum_{i=1}^{\infty} \lambda_i^{-1},$$

and let

$$\Gamma_h = \sum_{i=1}^{N-1} \lambda_{h,i}^{-1}.$$

Proposition 6.2. *There exists a constant* $C > 0$, *independent of* h, *so that*

$$|\Gamma - \Gamma_h| \le Ch^4. \tag{6.8}$$

Proof. We introduce the (infinite) set of reciprocals of the eigenvalues of $\mathcal{L}$, namely, the eigenvalues of the kernel $K(x, y)$ (5.3),

$$\Lambda^{-1} = \left\{\lambda_1^{-1} > \lambda_2^{-1} > \ldots > \lambda_k^{-1} \ldots > 0\right\}, \tag{6.9}$$

while

$$\Lambda_h^{-1} = \left\{\lambda_{h,1}^{-1} \ge \lambda_{h,2}^{-1} \ge \ldots \ge \lambda_{h,N-1}^{-1} > 0\right\} \tag{6.10}$$

is the set of eigenvalues of $(\delta_x^4)^{-1}$, corresponding to the discrete kernel K^h (5.6).

By the standard trace formula, it follows that

$$\Gamma = \int_0^1 K(x, x)dx, \quad \Gamma_h = h\sum_{i=1}^{N-1} K(x_i, x_i). \tag{6.11}$$

Since $K(x, x) = \frac{1}{3}x^3(1-x)^3$, the numerical values of Γ and C can easily be calculated, and it turns out that

$$\Gamma = \frac{1}{420}. \tag{6.12}$$

On the other hand

$$\Gamma_h = \frac{h}{3}\sum_{i=1}^{N-1}(ih)^3(1-ih)^3 = \frac{1}{420} + \frac{1}{180}h^4 - \frac{1}{126}h^6, \tag{6.13}$$

so that (6.8) is established (and even with an explicit constant). □

Remark 6.3. *Observe that Γ_h is the discrete trapezoidal approximation to the integral for Γ. By the standard estimate for the trapezoidal rule, we obtain*

$$|\Gamma - \Gamma_h| \le Ch^2, \tag{6.14}$$

with $C = \frac{1}{12}\max_{0\le x\le 1}|(\frac{d}{dx})^2 K(x,x)| = \frac{1}{96}$.

The fourth-order estimate (6.8) *is clearly a result of special properties of the kernel K.*

The "collective" estimate (6.8) does not imply that an estimate of the form $\lambda_i^{-1} - \lambda_{h,i}^{-1} = O(h^4)$ is valid, for any fixed value of the index i. However, the next proposition provides a weaker statement in this direction. It plays a key role in the final, stronger Theorem 6.13 below.

Proposition 6.4. *For any fixed integer $i \ge 1$ there exist positive constants C, $h_0 > 0$ such that for any $0 < h = \frac{1}{N} < h_0$ we have*

$$dist\{\lambda_i^{-1}, \Lambda_h^{-1}\} \le Ch^4, \tag{6.15}$$

where Λ_h^{-1} is the set of reciprocals introduced in (6.10).

Proof. Let $\phi_i(x) \in H_0^2(\Omega)$ be a normalized eigenfunction of $\left(\frac{d}{dx}\right)^4$, corresponding to λ_i. Recall that $\phi_i \in C^\infty$ and $\left(\frac{d}{dx}\right)^{-4}\phi_i = \lambda_i^{-1}\phi_i$. Hence

$$\lambda_i^{-1}\phi_i(x) = \int_0^1 K(x,y)\phi_i(y)dy, \quad x \in \Omega.$$

For simplicity, we denote by $\{x_j = jh,\ 0 \le j \le N\}$ the grid points , omitting the obvious dependence on h.

Let $\phi_i^* = \{\phi_i(x_0), \dots, \phi_i(x_k), \dots, \phi_i(x_N)\}$ be the corresponding grid function.

In view of Claim 2.3 and Corollary 5.2 we have for all $0 \le k \le N$,

$$\left|\lambda_i^{-1}\phi_i(x_k) - h\sum_{j=0}^{N} K(x_k, x_j)\phi_i(x_j)\right| \le Ch^4,$$

where here and below $C > 0$ is a constant depending only on ϕ_i that changes from one estimate to the next. Using the notation (5.6) this can be rewritten as

$$\left|\lambda_i^{-1}\phi_i^*(x_k) - \sum_{j=0}^{N} K_{k,j}^h\phi_i^*(x_j)\right| \le Ch^4, \tag{6.16}$$

that is

$$\left|(\lambda_i^{-1} - (\delta_x^4)^{-1})\phi_i^*\right|_h \le Ch^4.$$

On the other hand, the smoothness of the normalized ϕ_i yields

$$|\phi_i^*|_h \ge 1 - Ch.$$

The last two estimates imply the following estimate of the operator norm,

$$\left|\left(\lambda_i^{-1} - (\delta_x^4)^{-1}\right)^{-1}\right|_h \ge \frac{1-Ch}{Ch^4} \ge Ch^{-4}, \tag{6.17}$$

for $h < h_0$. A standard result concerning resolvents of self-adjoint operators now yields

$$dist\{\lambda_i^{-1}, \Lambda_h^{-1}\} = \left|\left(\lambda_i^{-1} - (\delta_x^4)^{-1}\right)^{-1}\right|_h^{-1},$$

which concludes the proof of the proposition. □

Remark 6.5. *Proposition 6.4 shows that in any neighborhood of λ_i^{-1} there is a discrete eigenvalue $\lambda_{h,k}^{-1}$, provided $h>0$ is sufficiently small. Observe, however, that we cannot infer that, even the largest eigenvalue (of $\mathcal{L}^{-1}$) λ_1^{-1} is the limit, as $h\to 0$, of the largest discrete eigenvalue $\lambda_{h,1}^{-1}$ (of $(\delta_x^4)^{-1}$).*

Remark 6.6. *In view of Corollary 5.2 the discrete eigenvalues in Λ_h^{-1} are obtained by a "Nyström method"* [26], *namely, eigenvalues of the discretized kernel. The fact that for any fixed integer $i\geq 1$*

$$\lim_{h\to 0} dist\{\lambda_i^{-1},\Lambda_h^{-1}\}=0,$$

follows from [26, Theorem 3]. *Proposition 6.4 establishes an "optimal" $O(h^4)$ rate to this convergence.*

6.3. **THE FIRST EIGENVALUE.** Our goal is the convergence of individual eigenvalues. We begin with a general discussion and its implication for the first eigenvalue.

Pick $\phi\in\{\phi_1,...,\phi_k...\}$ a normalized eigenfunction of $\mathcal{L}$, with associated eigenvalue $\lambda\in\{0<\lambda_1<\lambda_2<\ldots<\lambda_k<\ldots\}$.

Applying the operator $\mathcal{L}$ to

$$\mathcal{L}\phi=\lambda\phi,\quad \lambda\in\{0<\lambda_1<...<\lambda_k...\}..,$$

we get

$$\frac{d^8}{dx^8}\phi=\lambda^2\phi.$$

Since ϕ is normalized, we have

$$\|\frac{d^8}{dx^8}\phi\|_{L^2[0,1]}=\lambda^2, \tag{6.18}$$

and continuing in this fashion we see that all derivatives of ϕ are bounded by some power of λ, and therefore in the estimates below we have a generic constant $C>0$ depending only on λ.

Let ϕ^* be the corresponding grid function, $\phi^*(x_i)=\phi(x_i),\quad 0\leq i\leq N.$

Let $\mathfrak{v}\in l^2_{h,0}$ satisfy

$$\delta_x^4\mathfrak{v}=\lambda\phi^*,$$

where also $\mathfrak{v}_x\in l^2_{h,0}$.

By the fourth order accuracy (2.15) we know

$$|\mathfrak{v}-\phi^*|_\infty\leq Ch^4, \tag{6.19}$$

where C is independent of $N=h^{-1}$, but depends of course on ϕ.

It follows that

$$\delta_x^4\mathfrak{v}=\lambda\mathfrak{v}+\mathfrak{w},\ |\mathfrak{w}|_h\leq Ch^4. \tag{6.20}$$

Since ϕ is normalized, the truncation error for the trapezoid integration gives

$$|\phi^*|_h^2=h\sum_{i=1}^{N-1}[\phi_i^*]^2=\|\phi\|^2_{L^2[0,1]}+O(h^2)=1+O(h^2), \tag{6.21}$$

hence also

$$|1-|\mathfrak{v}|_h^2|\leq Ch^2. \tag{6.22}$$

Let $\bar{\mathfrak{v}}=\frac{\mathfrak{v}}{|\mathfrak{v}|_h}$, then it follows from (6.20)

$$\delta_x^4\bar{\mathfrak{v}}=\lambda\bar{\mathfrak{v}}+\bar{\mathfrak{w}},\ |\bar{\mathfrak{w}}|_h\leq Ch^4. \tag{6.23}$$

Regarding the first eigenvalue, we can now show that $\lambda_{h,1}$ can exceed λ_1 by at most $O(h^4)$.

Claim 6.7. *Let λ_1 be the first eigenvalue of $\mathcal{L}$ (by* (6.7)*, $\lambda_1=\beta_0^4$). Then there exists a constant $C>0$, depending on the eigenfunction ϕ_1, but not on h, such that*

$$\lambda_{h,1}\leq\lambda_1+Ch^4. \tag{6.24}$$

Proof. Consider (6.23) with $\lambda = \lambda_1$. By the variational minimum principle for the first eigenvalue we know that

$$\lambda_{h,1} = \min_{|\mathfrak{z}|_h=1} (\delta_x^4 \mathfrak{z}, \mathfrak{z})_h,$$

hence

$$\lambda_{h,1} \leq (\delta_x^4 \bar{\mathfrak{v}}, \bar{\mathfrak{v}})_h \leq \lambda_1 + Ch^4, \tag{6.25}$$

which proves the claim. □

Remark 6.8. *The exact first eigenvalue is $\lambda_1 = 500.5639017404$. Numerical calculations actually show that $\lambda_{h,1} \leq \lambda_1$, and that $\lambda_{h,1}$ increases as h decreases. This is shown in Figure 1. It remains an open problem.*

Remark 6.9. *Observe that in Claim 6.7 we do not have a corresponding lower limit, namely, that $\lambda_{h,1}$ is above $\lambda_1 - O(h^4)$. This is evident in the numerical results displayed in Figure 2. The proof of this fact is postponed to Theorem 6.13 below, where we show that the convergence of all discrete eigenvalues to the corresponding continuous ones is "optimal", namely, at an $O(h^4)$ rate.*

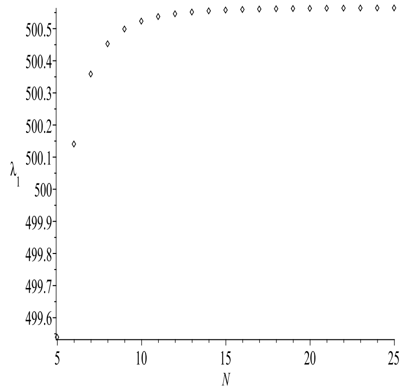

FIGURE 1. First discrete eigenvalue as a function of the number of grid points in $[0, 1]$.

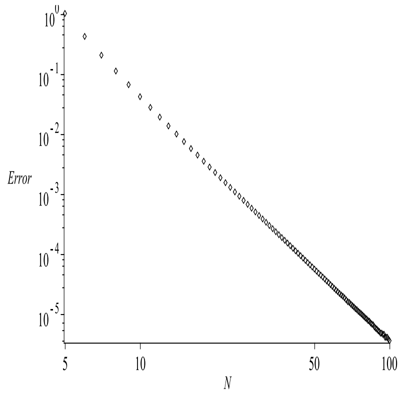

FIGURE 2. Log-log graph of the error of first discrete eigenvalue $\lambda_1 - \lambda_{h,1}$ as function of the number N of grid points in $[0, 1]$. The slope is -4, indicating a convergence rate $O(N^{-4}) = O(h^4)$.

6.4. **CONVERGENCE OF THE DISCRETE EIGENVALUES** $\lambda_{h,k},\ k \geq 1$.

We now consider the convergence of all discrete eigenvalues to their continuous counterparts.

Numerical simulations indicate that, if we *fix an index* k, then

$$|\lambda_k - \lambda_{h,k}| \leq Ch^4, \quad \text{as } h \to 0,$$

with $C > 0$ depending on k. This is demonstrated in Figure 3 (for $N = 16$) and Figure 4 (for $N = 64$). We thank Jean-Pierre Croisille for both figures. Thus, even a very coarse resolution produces excellent approximation of the eigenvalues.

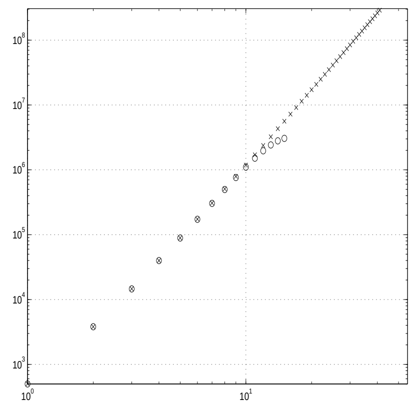

FIGURE 3. Graph of eigenvalues in logarithmic scale: k−− Horizontal, $\log \lambda_k$ $(\times)$, $\log \lambda_{h,k}$ $(\circ)$, $h = \frac{1}{N} = \frac{1}{16}$ −−Vertical .

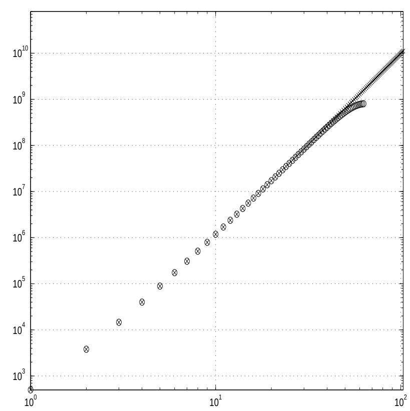

FIGURE 4. Graph of eigenvalues in logarithmic scale: k−− Horizontal, $\log \lambda_k$ $(\times)$, $\log \lambda_{h,k}$ $(\circ)$, $h = \frac{1}{N} = \frac{1}{64}$ −−Vertical .

In dealing with the convergence of *all eigenvalues*, it seems that one cannot employ an approach based on the Rayleigh-Ritz quotient. We review here a very different approach,that will yield not only the convergence to all eigenvalues, but, furthermore, the optimal $O(h^4)$ convergence rate.

We introduce a piecewise constant kernel $K_h(x, y)$ by

$$K_h(x, y) = K(x_i, y_j), \quad x \in \Big(x_i - \frac{h}{2}, x_i + \frac{h}{2}\Big),\ y \in \Big(y_j - \frac{h}{2}, y_j + \frac{h}{2}\Big),\ 0 \leq i, j \leq N. \tag{6.26}$$

Clearly for $i = 0$ the endpoint $x_0 - \frac{h}{2}$ is replaced by x_0, and similarly for the other endpoints.

We denote by $\mathcal{L}_h^{-1}$ the operator (on $L^2[0,1]$) whose kernel is K_h. Clearly this operator is compact and positive definite. In fact, the following claim asserts that it has only finitely many positive eigenvalues (depending on h, of course).

Claim 6.10. *The set of eigenvalues of $\mathcal{L}_h^{-1}$ is the finite set Λ_h^{-1}, defined in* (6.10).

Proof. Let $u \in L^2[0,1]$ be an eigenfunction of $\mathcal{L}_h^{-1}$. Thus, for some $\mu > 0$,

$$\mu u(x) = \int_0^1 K_h(x,y)u(y)dy, \quad x \in [0,1].$$

In particular, u is piecewise constant

$$u(x) = u(x_i), \quad x \in \Big(x_i - \frac{h}{2}, x_i + \frac{h}{2}\Big),\ i = 0,1,\ldots,i = N.$$

hence (with K^h as in Corollary 5.2)

$$\mu u(x_i) = \sum_{j=0}^{N} K^h_{i,j} u(x_j), \quad 0 \le i \le N, \tag{6.27}$$

where the boundary values $u(x_0) = u(x_N) = 0$ are included.

Thus μ is an eigenvalue of $(\delta_x^4)^{-1}$, hence $\mu = \lambda_{h,k}^{-1}$ for some $1 \le k \le N-1$. □

We now proceed to establish the convergence of all discrete eigenvalues to the corresponding continuous ones. In fact, the following lemma is a special case of a theorem of Markus [20, Corollary 5.3] concerning differences of eigenvalues of self-adjoint operators. A similar general theorem was proved (much later) by Kato [18]. However the generality of Kato's theorem required an "extended enumeration" of the eigenvalues, adding values of boundary points of the essential spectra.

For the convenience of the reader we provide here a simple proof of the lemma, following the proof of (the finite-dimensional) Theorem 6.11 in [17, Section II.6].

Lemma 6.11. *Let $h = \frac{1}{N}$, and let*

$$\Lambda^{-1} = \left\{\lambda_1^{-1} > \lambda_2^{-1} > \ldots > \lambda_k^{-1} \ldots > 0\right\},$$

$$\Lambda_h^{-1} = \left\{\lambda_{h,1}^{-1} \ge \lambda_{h,2}^{-1} \ge \ldots \ge \lambda_{h,N-1}^{-1} > 0\right\},$$

be the sets introduced in (6.9), (6.10)*, respectively.*

Then there exists a constant $C > 0$, independent of h, so that

$$\sum_{k=1}^{N-1} |\lambda_k^{-1} - \lambda_{h,k}^{-1}|^2 + \sum_{k=N}^{\infty} \lambda_k^{-2} \le \int_0^1\int_0^1 |K(x,y) - K_h(x,y)|^2 dxdy \le Ch^2. \tag{6.28}$$

Proof. Note that both $\mathcal{L}^{-1}$, $\mathcal{L}_h^{-1}$, are Hilbert-Schmidt (hence compact) positive operators.

For $t \in [0,1]$ let

$$\mathcal{L}_{t,h}^{-1} = (1-t)\mathcal{L}^{-1} + t\mathcal{L}_h^{-1},$$

which is also a compact, positive self-adjoint operator. In particular, its spectrum (apart from 0) consists of a descending sequence of positive eigenvalues

$$\left\{\mu_1^{-1}(t) \ge \mu_2^{-1}(t) \ge \ldots \ge \mu_{N-1}^{-1}(t) \ge \mu_N^{-1}(t) \ge \ldots \mu_{N+p}^{-1}(t) \ge \ldots > 0\right\}, 0 \le t \le 1.$$

In view of the discussion in [17, Chapter VII.3.2] the functions $\mu_k^{-1}(t)$, $1 \le k < \infty$, are continuous, piecewise analytic functions of t, and satisfy

$$\mu_k^{-1}(0) = \lambda_k^{-1},\ 1 \le k < \infty, \tag{6.29}$$

and

$$\mu_k^{-1}(1) = \begin{cases} \lambda_{h,k}^{-1},\ 1 \le k < N, \\ 0,\ k \ge N. \end{cases} \tag{6.30}$$

In addition, there exists (for every fixed $t \in [0,1]$) a corresponding set of orthonormal functions (in $L^2(0,1)$)

$$\{\phi_1(x;t), \phi_2(x;t), \dots, \phi_N(x;t), \dots, \phi_k(x;t), \dots\}, \; 0 \le t \le 1.$$

Pick an index $k \ge 1$. The eigenvalue $\mu_k^{-1}(t)$ is continuous (in $t \in [0,1]$) and piecewise analytic, with finitely many singularities. The associated eigenfunction $\phi_k(x;t)$ is piecewise analytic in t, with the same (finitely many) singularities. Thus, the equation

$$\Big[(1-t)\mathcal{L}^{-1} + t\mathcal{L}_h^{-1} - \mu_k^{-1}(t)\Big]\phi_k(x;t) = 0 \tag{6.31}$$

can be differentiated with respect to t (excluding the singularities) and we obtain

$$\Big[\mathcal{L}_h^{-1} - \mathcal{L}^{-1} - \frac{d}{dt}\mu_k^{-1}(t)\Big]\phi_k(x;t) + \Big[(1-t)\mathcal{L}^{-1} + t\mathcal{L}_h^{-1} - \mu_k^{-1}(t)\Big]\frac{d}{dt}\phi_k(x;t) = 0. \tag{6.32}$$

Taking the scalar product with $\phi_k(x;t)$ we conclude that

$$\frac{d}{dt}\mu_k^{-1}(t) = \Big((\mathcal{L}_h^{-1} - \mathcal{L}^{-1})\phi_k(x;t), \phi_k(x;t)\Big)_{L^2(0,1)}, \quad t \in [0,1]. \tag{6.33}$$

Integrating this equation and taking (6.29) and (6.30) into account we get

$$\int_0^1 \Big((\mathcal{L}_h^{-1} - \mathcal{L}^{-1})\phi_k(x;t), \phi_k(x;t)\Big)_{L^2(0,1)} dt = \begin{cases} \lambda_{h,k}^{-1} - \lambda_k^{-1}, \; 1 \le k < N, \\ \\ -\lambda_k^{-1}, \; k \ge N. \end{cases} \tag{6.34}$$

The self-adjoint operator $\mathcal{A} = \mathcal{L}_h^{-1} - \mathcal{L}^{-1}$ is Hilbert-Schmidt, hence compact. Let $\{\gamma_1, \gamma_2, \dots\}$ be the sequence of its non-zero eigenvalues (repeated according to multiplicity) with a corresponding orthonormal sequence of eigenfunctions $\{\chi_1(x), \chi_2(x), \dots\} \subseteq L^2(0,1)$.

Since $\phi_k(x;t) = \sum\limits_{j=1}^{\infty}(\phi_k(x;t), \chi_j(x))_{L^2(0,1)}\chi_j(x)$, Equation (6.34) entails

$$\sum_{j=1}^{\infty}\sigma_{j,k}\gamma_j = \begin{cases} \lambda_{h,k}^{-1} - \lambda_k^{-1}, \; 1 \le k < N, \\ \\ -\lambda_k^{-1}, \; k \ge N, \end{cases} \tag{6.35}$$

where $\sigma_{j,k} = \int_0^1 (\phi_k(x;t), \chi_j)^2_{L^2(0,1)} dt, \; 1 \le j,k < \infty$.

By the orthonormality of the functions (in x)

$$0 \le \sigma_{j,k} \le 1, \quad \sum_{j=1}^{\infty}\sigma_{j,k} \le 1, \quad \sum_{k=1}^{\infty}\sigma_{j,k} \le 1.$$

Let Φ be a real convex function on the real line, with $\Phi(0) = 0$. From Jensen's inequality we get

$$\Phi\Big(\sum_{j=1}^{\infty}\sigma_{j,k}\gamma_j\Big) \le \sum_{j=1}^{\infty}\sigma_{j,k}\Phi(\gamma_j), \quad k = 1, 2, \dots,$$

and summation over k yields

$$\sum_{k=1}^{\infty}\Phi\Big(\sum_{j=1}^{\infty}\sigma_{j,k}\gamma_j\Big) \le \sum_{j=1}^{\infty}\Phi(\gamma_j). \tag{6.36}$$

In particular , taking $\Phi(\xi) = \xi^2$ and noting (6.35) we obtain

$$\sum_{k=1}^{N-1}|\lambda_k^{-1} - \lambda_{h,k}^{-1}|^2 + \sum_{k=N}^{\infty}\lambda_k^{-2} \le \sum_{j=1}^{\infty}\gamma_j^2.$$

The sum on the right-hand side is the square of the Hilbert-Schmidt norm of $\mathcal{A}$, which is $\int_0^1\int_0^1 |K(x,y) - K_h(x,y)|^2 dxdy$, thus proving (6.28).

$\square$

	k=1	*k=2*	*k=3*	*k=4*
true eigenvalue	*500.563902*	*3803.537080*	*14617.630131*	*39943.799006*
N=10	*500.521885*	*3800.689969*	*14567.617771*	*39493.816015*
N=20	*500.561614*	*3803.398598*	*14615.468848*	*39926.599754*
N=30	*500.563462*	*3803.511145*	*14617.236978*	*39940.722654*
N=40	*500.563764*	*3803.529031*	*14617.509451*	*39942.881883*
N=50	*500.563845*	*3803.533813*	*14617.581402*	*39943.430972*
N=60	*500.563874*	*3803.535512*	*14617.606815*	*39943.623511*

TABLE 1. First 4 eigenvalues (top row) and their numerical approximations using a grid of $N = 10 - 60$ nodes.

Remark 6.12. *Note that* (6.28) *yields in particular the uniform estimate*

$$\sum_{k=1}^{N-1} |\lambda_k^{-1} - \lambda_{h,k}^{-1}|^2 \leq Ch^2. \tag{6.37}$$

This estimate is valid simultaneously for all $N-1$ eigenvalues. Fixing an index k, we get in particular

$$\frac{|\lambda_k - \lambda_{h,k}|}{\lambda_{h,k}} \leq C\lambda_k h. \tag{6.38}$$

In view of Claim 6.1 we have $\lambda_k \approx k^4$. Thus (6.38) *yields only an $O(h)$ convergence.*

However it is seen in Table 1, that even with a small number of grid points, the first discrete eigenvalues approximate very well the continuous ones. We prove in Theorem 6.13 below that the convergence is indeed "optimal".

We now proceed to prove the "optimal" estimate.

Theorem 6.13 (Optimal rate of convergence of discrete eigenvalues). *Fix an integer $k \geq 1$ and consider the discrete eigenvalue $\lambda_{h,k}$ as a function of $h = \frac{1}{N}$, $N = k+1, k+2, \dots$. Then there exists a constant $C > 0$, depending only on k, such that*

$$|\lambda_k - \lambda_{h,k}| \leq Ch^4, \quad 0 < h < h_0. \tag{6.39}$$

Proof. Fix k. If $j \neq k$ then we have, by (6.37)

$$|\lambda_{h,j}^{-1} - \lambda_k^{-1}| \geq \left|\lambda_j^{-1} - \lambda_k^{-1}\right| - \left|\lambda_j^{-1} - \lambda_{h,j}^{-1}\right| \geq \left|\lambda_j^{-1} - \lambda_k^{-1}\right| - C^{\frac{1}{2}}h.$$

Therefore, if $\eta = \min_{j\neq k} \left|\lambda_j^{-1} - \lambda_k^{-1}\right|$, then for $h < h_0 = \frac{1}{2}\eta C^{-\frac{1}{2}}$ we have

$$j \neq k \quad \Rightarrow \quad |\lambda_{h,j}^{-1} - \lambda_k^{-1}| \geq \frac{1}{2}\eta.$$

Combined with Proposition 6.4 we infer that the only element of Λ_h^{-1} that can be "close" to λ_k^{-1} is $\lambda_{h,k}^{-1}$, and that

$$|\lambda_k^{-1} - \lambda_{h,k}^{-1}| \leq Ch^4,$$

thus concluding the proof of the theorem. □

Remark 6.14. *Observe that in the proof of Theorem 6.13 we relied on special properties of the kernel, via Proposition 6.4. Without using such information we obtain "sub-optimal" estimates. For example,* (6.28) *implies*

$$\sum_{k=N}^{\infty} \lambda_k^{-2} \leq CN^{-2},$$

which is not optimal, in view of Claim 6.1. Compare also to the estimate in (6.8) *which can be written as*

$$|\sum_{i=1}^{\infty} \lambda_i^{-1} - \sum_{i=1}^{N-1} \lambda_{h,i}^{-1}| \leq Ch^4.$$

Remark 6.15. *The $O(h^4)$ rate of convergence, as stated in Theorem 6.13, can be compared to the method of collocation approximation* [12]. *In the case of the latter , achieving a similar rate of convergence requires the construction of an interpolating C^3 piecewise fifth-order polynomial function, and then using collocation at Gaussian points. The results here were obtained by using the discretized kernel (of the inverse operator). Owing to the observed connection between this kernel and the classical (C^2) cubic splines, the approximating eigenvalues are in fact those of the fourth-order (distributional) derivative of the interpolating cubic spline at the grid points (Proposition 3.8).*

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